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A NOVEL DERIVATION OF NON-HYPERSINGULAR TIME-DOMAIN BIEs FOR TRANSIENT ELASTODYNAMIC CRACK ANALYSIS

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Abstract—A novel application of conservation integrals in transient elastodynamic crack analysis is presented to derive non-hypersingular time-domain boundary integral equations (BIEs). The new derivation is based on an elastodynamic conservation integral which is employed to express the displacement gradients in terms of an integral over the surface of the crack. BIEs are obtained by substituting this representation integral into Hooke's law and by taking a limit process. The BIEs obtained in this manner are valid for arbitrary crack configurations, and they are immediately non-hypersingular. The Laplace and the Fourier transforms of these BIEs agree with those obtained by other authors by using the conventional derivation in conjunction with regularization techniques. Numerical examples show that the time-domain BIEs presented here can yield highly accurate results for elastodynamic stress intensity factors.

1. INTRODUCTION

Conservation laws or path-independent integrals are frequently used for characterizing the singular stress and strain fields near crack tips in fracture mechanics. The most widely used path-independent integral in fracture mechanics is the J integral of Eshelby (1956) and Rice (1968). In linear elastic fracture mechanics, the J integral has a precise and clear physical interpretation as the energy release rate per unit crack-tip extension, and it is related to the well-known stress intensity factors which control the singular crack-tip field. In elastic–plastic fracture mechanics under the use of the deformation theory of plasticity the value of J represents the strength of the HRR-singularity field which dominates in a region larger than the fracture process zone and the zone of finite deformation. Thus, in both linear and non-linear fracture mechanics the onset of crack growth can be correlated with a critical value of J. The attractive features of J, namely its energy-based definition and its path independence have been advantageously exploited in the development of fracture mechanics. In recent years, several new conservation integrals have been proposed as crack-tip characterizing parameters, to take thermal, inertial and inelastic effects into account. Most of these works are motivated by the belief that new conservation integrals will lead to further advances in fracture mechanics. In this paper, a novel application of conservation integrals in transient elastodynamic crack analysis is presented, to derive suitable time-domain BIEs. The corresponding frequency-domain formulation has recently been presented by Zhang and Achenbach (1989b).

The conventional derivation of time-domain BIEs is an extension of the procedure for elastostatics proposed by Rizzo (1967) and Cruse (1969). This derivation is based on the Betti–Rayleigh reciprocal theorem for two independent elastodynamic states of the same body. By choosing one of the states as the unknown scattered field and the other as the fundamental solution (the Green's function) due to an impulsive unit point force, a representation integral for the scattered displacement field can be derived. For wave scattering by a crack in an unbounded body, the integral is over the surface of the crack and it contains the crack-opening displacements (the displacement jumps across the crack faces) and the Green's function terms in its integrand. The usual limiting process on this representation integral as the observation point approaches the crack faces leads to a degenerate BIE formulation, as shown by Cruse (1978) for elastostatic crack analysis. This difficulty is overcome by the use of the representation integral for the traction components, and their corresponding boundary integral equations. Such BIEs are, however, hypersingular, and

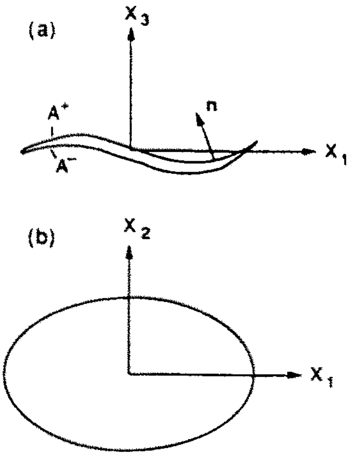

Fig. 1. A 3D crack of arbitrary shape: (a) x_1x_3-plane, (b) top view.

they cannot be solved directly by numerical methods. To circumvent these difficulties several regularization procedures have been proposed (Budiansky and Rice, 1979; Budreck and Achenbach, 1988; Guo *et al.*, 1988; Hirose and Achenbach, 1988, 1989; Nishimura *et al.*, 1987a,b, 1988; Nishimura and Kobayashi, 1988, 1989; Schmerr, 1982; Sládek and Sládek, 1984; Tan, 1975; Zhang and Achenbach, 1988a,b, 1989a; Zhang, 1990a). Most of these works first reduce the higher-order singularities to integrable ones, and then solve the modified BIEs numerically. The reduction of the hypersingularities is achieved by using partial integration techniques. The required manipulations are reasonably easy for simple crack configurations such as 3D planar or 2D straight cracks, but they become cumbersome for arbitrarily-shaped cracks.

This paper presents a novel derivation of non-hypersingular time-domain BIEs for transient elastodynamic crack analysis. The new derivation is based on an elastodynamic conservation integral, the I_k integral. By using this integral, a two-state conservation integral is derived which can be employed to express the gradients of the scattered displacements in terms of an integral over the surface of the crack. The representation integral relates the scattered displacements and their derivatives via Green's function terms in the integrand. The corresponding representation integral for the traction components is derived by the use of Hooke's law. BIEs are subsequently obtained by taking a limit process on the representation formula for the tractions. The BIEs that are obtained in this manner are valid for arbitrary crack configurations, and their Fourier and Laplace transforms agree with those results obtained by other authors in the transformed domain, via the conventional derivation in conjunction with regularization techniques. An essential advantage of the new derivation is that it leads directly to non-hypersingular BIEs, which allow an immediate and reliable numerical implementation. BIEs derived from the complementary conservation integral I_k^c are also given, but these equations do not offer advantages over the conventional formulation.

2. GOVERNING EQUATIONS AND CONVENTIONAL BIE DERIVATION

Let us consider a three-dimensional crack of arbitrary shape in an infinite, homogeneous, isotropic and linearly elastic solid, as shown in Fig. 1. The faces of the crack are assumed to be infinitesimally close prior to loading, and they do not interact with each other when external loads are applied. This approximation is acceptable for real cracks whose faces are initially sufficiently separated so that the faces will not touch when the body is disturbed.

The stress equations of motion are given by (Achenbach, 1973; Eringen and Suhubi, 1975)

$$\sigma_{ij,j}+\rho f_i = \rho\ddot{u}_i, \tag{1}$$

where σ_{ij} denotes the stress components, f_i denotes the body force components, u_i defines the displacement components, and ρ is the mass density. In eqn (1), superscript dots indicate derivatives with respect to time t, $(*)_{,j}$ represents derivatives of $(*)$ with respect to spatial variables x_j, and the conventional summation rule over double indices is implied. In the linear theory the strain components are defined by

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}). \tag{2}$$

The stress and the strain components are related by Hooke's law

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \tag{3}$$

where C_{ijkl} are the components of the elasticity tensor which for isotropic materials can be written as

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}). \tag{4}$$

Here, λ and μ are Lamé's elastic constants and δ_{ij} is the Kronecker delta. As boundary conditions, the tractions vanish on the faces of the crack, i.e.

$$t_i = \sigma_{ij}n_j = 0, \quad \mathbf{x}\in A, \tag{5}$$

where $A = A^+ + A^-$. For a scattering problem A^+ is the insonified side of the crack and A^- is the shadow side. Also, n_j is the unit normal vector of A. The initial conditions are

$$u_i(\mathbf{x},t) = \dot{u}_i(\mathbf{x},t) = 0 \quad \text{for } t = 0. \tag{6}$$

Assume that the crack is subjected to a loading induced by an incident wave; then the total field generated by the interaction of the incident wave with the crack can be written as

$$u_i = u_i^{\text{in}}+u_i^{\text{sc}}, \quad \sigma_{ij} = \sigma_{ij}^{\text{in}}+\sigma_{ij}^{\text{sc}}, \tag{7}$$

where u_i^{in} and σ_{ij}^{in} represent the incident field in the absence of the crack, and u_i^{sc} and σ_{ij}^{sc} define the scattered field. For a given incident field, the scattered field has to be determined so that the governing equations (1)–(6) are satisfied for all values of time t.

The conventional derivation of time-domain BIEs is an extension of the procedure for elastostatics proposed by Rizzo (1967) and Cruse (1969). This derivation is based on the elastodynamic reciprocal theorem which relates two distinct elastodynamic states of the same body. By taking one state as the unknown scattered field and the other as the fundamental solution due to an impulsive unit point force, a representation integral for the scattered displacements can be obtained. For the present problem, this representation integral takes the following form

$$u_k^{\text{sc}}(\mathbf{x},t) = \int_{A^+}\sigma_{ijk}^{G} * \Delta u_i n_j \,\mathrm{d}A(\mathbf{y}), \quad \mathbf{x}\notin A^+, \tag{8}$$

where $\mathbf{x}$ and t denote the position vector and the time variable of the observation point, σ_{ijk}^{G} is the stress Green's function of the uncracked fullspace (Appendix A), and Δu_i defines the crack-opening displacements (displacement jumps across the faces of the crack). Also, $*$ denotes Riemann convolution

$$g * h(\mathbf{x}, t) = \int_0^t g(\mathbf{x}, t-\tau) h(\mathbf{x}, \tau)\, \mathrm{d}\tau. \tag{9}$$

Boundary integral equations can be derived from (8) by taking the limit $\mathbf{x} \to A^+$. However, such BIEs degenerate for crack analysis, as shown by Cruse (1978) for the static case. This difficulty is overcome by using the representation integral for the traction components, which is obtained by substituting eqn (8) into Hooke's law and by using $t_p = \sigma_{pq} n_q$ as

$$t_p^{sc}(\mathbf{x}, t) = -C_{pqkl} n_q(\mathbf{x}) \int_{A^+} \sigma^G_{ijk,l} * \Delta u_i n_j \, \mathrm{d}A(\mathbf{y}), \quad \mathbf{x} \notin A^+. \tag{10}$$

Then, BIEs can be derived from eqn (10) by letting $\mathbf{x} \to A^+$ and by applying boundary conditions (5). Unfortunately, these BIEs are hypersingular when the observation point $\mathbf{x}$ and the source point $\mathbf{y}$ coincide, since in this case the terms $\sigma^G_{ijk,l}$ behave as (Wheeler and Sternberg, 1968)

$$\sigma^G_{ijk,l} \propto \begin{cases} 1/r^2, & 2\mathrm{D}, \\ 1/r^3, & 3\mathrm{D}, \end{cases} \quad \text{as } \mathbf{r} \to 0, \tag{11}$$

where $r = |\mathbf{x}-\mathbf{y}|$. For two-dimensional problems, additional hypersingularities like

$$\sigma^G_{ijk,l} \propto \frac{1}{[(t-\tau)^2 - r^2/c_\alpha^2]^{3/2}}, \quad \text{as } (t-\tau)^2 \to r^2/c_\alpha^2, \tag{12}$$

occur in eqn (10). Here, c_α is either the longitudinal wave speed c_L or the transverse wave speed c_T, where

$$c_L = [(\lambda+2\mu)/\rho]^{1/2}, \quad c_T = (\mu/\rho)^{1/2}. \tag{13}$$

The hypersingularities (11) and (12) prevent a reliable direct numerical solution of the system of BIEs obtained from eqn (10). Thus, special attention must be paid in developing numerical procedures.

For cracks under transient loading, several regularization techniques have been proposed in the literature (Guo *et al.*, 1988; Hirose and Achenbach, 1988, 1989; Nishimura *et al.*, 1987a,b, 1988; Zhang and Achenbach, 1988a, 1989a; Zhang, 1990a). Similar investigations for cracks under time-harmonic wave loading can be found in Budiansky and Rice (1979), Budreck and Achenbach (1988), Nishimura and Kobayashi (1988, 1989), Schmerr (1982), Sládek and Sládek (1984), Tan (1975) and Zhang and Achenbach (1988b). The corresponding elastostatic crack analysis using BIE methods was presented several years ago by Cruse (1978), Weaver (1977) and many other authors (Cruse, 1988). The common feature of all these works is the use of partial integration to reduce the higher-order singularities. This technique is easily implemented and well established for flat or straight cracks, but it becomes cumbersome for cracks of arbitrary shape. In the following sections, a novel derivation is presented which is based on an elastodynamic conservation integral.

3. THE I_k INTEGRAL AND THE NOVEL BIE DERIVATION

As in elastostatics, several conservation laws or path-independent integrals are valid for transient elastodynamics (Fletcher, 1976; Gurtin, 1976, 1977; Jiang, 1986; Zhang, 1990b). One of them may be stated as

$$I_k = \int_S [\tfrac{1}{2}(\sigma_{mn} * u_{m,n} + \rho\ddot{u}_i * u_i)\delta_{jk} - \sigma_{ij} * u_{i,k}]n_j \, dS - \int_V \rho f_i * u_{i,k} \, dV = 0, \tag{14}$$

where S is the surface and V is the volume of a body, and n_j is the unit outward normal vector. The assumptions for (14) to be true are the absence of singularities in V and the null initial conditions, eqn (6). To prove eqn (14), the divergence theorem is applied to the first integral of (14). This yields

$$I_k = \int_V [\tfrac{1}{2}(\sigma_{mn} * u_{m,n} + \rho\ddot{u}_i * u_i)_{,k} - (\sigma_{ij} * u_{i,k})_{,j}] \, dV - \int_V \rho f_i * u_{i,k} \, dV = 0. \tag{15}$$

By using the following properties of Riemann convolutions

$$\begin{aligned} g * h &= h * g, \\ (g+h) * e &= g * e + h * e, \\ (g * h)_{,l} &= g_{,l} * h + g * h_{,l}, \\ (g * h)^{\cdot} &= \dot{g} * h + g(\mathbf{x}, 0)h(\mathbf{x}, t), \end{aligned} \tag{16a–d}$$

it can be shown that

$$(\sigma_{ij} * u_{i,k})_{,j} = \tfrac{1}{2}(\sigma_{mn} * u_{m,n} + \rho\ddot{u}_i * u_i)_{,k} - \rho f_i * u_{i,k}, \tag{17}$$

where eqns (1)–(4) and (6) have been employed. Substitution of eqn (17) into eqn (15) leads to $I_k = 0$. This completes the proof of eqn (14).

Consider now two independent elastodynamic states of the same body

$$\{u_i^{(1)}, \sigma_{ij}^{(1)}, f_i^{(1)}\}, \tag{18}$$

$$\{u_i^{(2)}, \sigma_{ij}^{(2)}, f_i^{(2)}\}, \tag{19}$$

and require that these states satisfy the governing eqns (1)–(4) and the zero initial conditions (6). According to the superposition principle, the sum of (18) and (19)

$$u_i = u_i^{(1)} + u_i^{(2)}, \quad \sigma_{ij} = \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}, \quad f_i = f_i^{(1)} + f_i^{(2)}, \tag{20}$$

also satisfies eqns (1)–(4) and the zero initial conditions (6). Substitution of eqn (20) into (14) results in

$$I_k[u_i] = I_k[u_i^{(1)}] + I_k[u_i^{(2)}] + \int_S [(\sigma_{mn}^{(2)} * u_{m,n}^{(1)} + \rho\ddot{u}_i^{(1)} * u_i^{(2)})\delta_{jk} - \sigma_{ij}^{(2)} * u_{i,k}^{(1)} - \sigma_{ij}^{(1)} * u_{i,k}^{(2)}]n_j \, dS$$
$$- \rho \int_V (f_i^{(1)} * u_{i,k}^{(2)} + f_i^{(2)} * u_{i,k}^{(1)}) \, dV = 0. \tag{21}$$

Clearly, the terms $I_k[u_i^{(1)}]$ and $I_k[u_i^{(2)}]$ are identically zero because $u_i^{(1)}$ and $u_i^{(2)}$ are two distinct elastodynamic states which satisfy eqns (1)–(4) and the initial conditions (6). From eqn (21) the following identity is obtained

$$\int_S [(\sigma_{mn}^{(2)} * u_{m,n}^{(1)} + \rho\ddot{u}_i^{(1)} * u_i^{(2)})\delta_{jk} - \sigma_{ij}^{(2)} * u_{i,k}^{(1)} - \sigma_{ij}^{(1)} * u_{i,k}^{(2)}]n_j \, dS$$
$$= \rho \int_V (f_i^{(1)} * u_{i,k}^{(2)} + f_i^{(2)} * u_{i,k}^{(1)}) \, dV. \tag{22}$$

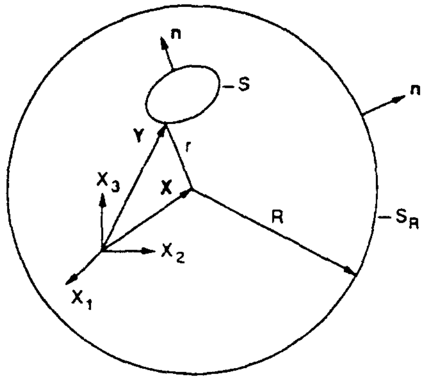

Fig. 2. A scatterer in an unbounded solid.

For $f_i^{(1)} = f_i^{(2)} = 0$, eqn (22) is reduced to the conservation law given by Gurtin (1976). The first state is now chosen to be the unknown field

$$\{u_i^{(1)}, \sigma_{ij}^{(1)}, f_i^{(1)}\} = \{u_i^{sc}, \sigma_{ij}^{sc}, 0\}, \tag{23}$$

while the second state is selected as the fundamental solution due to an impulsive unit point force

$$\{u_i^{(2)}, \sigma_{ij}^{(2)}, f_i^{(2)}\} = \{u_{il}^G a_l, \sigma_{ijl}^G a_l, \delta(\mathbf{x}-\mathbf{y})\delta(t) a_i\}, \tag{24}$$

where u_{il}^G and σ_{ijl}^G are Green's functions for the uncracked fullspace (see Appendix A), and a_l indicates the directions of the applied point force. By substituting (23) and (24) into eqn (22) and by using the sifting property of the delta function, the volume integral of eqn (22) is evaluated as

$$\rho \int_V (f_i^{(1)} * u_{i,k}^{(2)} + f_i^{(2)} * u_{i,k}^{(1)})\,\mathrm{d}V = a_l u_{l,k}^{sc}(\mathbf{x}, t). \tag{25}$$

Thus, eqn (22) can be rewritten as

$$u_{l,k}^{sc}(\mathbf{x}, t) = \int_S [(\sigma_{mnl}^G * u_{m,n}^{sc} + \rho u_{il}^G * \ddot{u}_i^{sc})\delta_{jk} - \sigma_{ijl}^G * u_{i,k}^{sc} - u_{il,k}^G * \sigma_{ij}^{sc}] n_j \,\mathrm{d}S, \quad \mathbf{x} \notin S, \tag{26}$$

in which $\mathbf{x}$ represents the position vector of the observation point, and $\mathbf{y}$ represents the position vector of the source point. Application of eqn (26) to the surfaces S and S_R (see Fig. 2) results in

$$u_{l,k}^{sc} = \int_S I_{lk}\,\mathrm{d}S + \int_{S_R} I_{lk}\,\mathrm{d}S, \tag{27}$$

where S is the surface of the scatterer, S_R is the surface of a sphere with radius R, centered at $\mathbf{x}$, and I_{lk} represents the integrand of (26). The surface S is assumed to be closed, regular and smooth, and the sphere with radius R must be sufficiently large so that the scatterer S and all singularities are contained in it. To examine the integral over the surface of the sphere, the following relationship can be used

$$r_{,k} = n_k, \tag{28}$$

$$\sigma^{G}_{mnl} * u^{sc}_{m,n} = \sigma^{G}_{mnl} n_n * \frac{\partial u^{sc}_m}{\partial r}, \tag{29}$$

$$dS = R^2\, d\Omega, \tag{30}$$

where n_k is the unit outward normal vector to the surface of the sphere and $d\Omega$ is an element of solid angle subtended by the area element dS. By substituting eqns (28)–(30) into (26) and (27), and by using the following asymptotic expressions of the Green's functions for $r \to \infty$ (see Appendix B)

$$\sigma^{G}_{mnl} n_n = -\rho \sum_{\xi = L,T} c_\xi \dot{u}^{G(\xi)}_{ml} + O(1/r^2), \tag{31}$$

$$u^{G}_{il,k} = -\sum_{\xi = L,T} \frac{1}{c_\xi} \dot{u}^{G(\xi)}_{il} n_k + O(1/r^2), \tag{32}$$

one obtains

$$\int_{S_R} I_{lk}\, dS = \int_{S_R} R^2 \left[\left(\rho \dot{u}^{sc}_i + \frac{1}{c_L} \dot{\sigma}^{sc}_{ij} n_j\right) * u^{G(L)}_{il}\right] n_k\, d\Omega$$
$$+ \int_{S_R} R^2 \left[\left(\rho \dot{u}^{sc}_i + \frac{1}{c_T} \dot{\sigma}^{sc}_{ij} n_j\right) * u^{G(T)}_{il}\right] n_k\, d\Omega. \tag{33}$$

Here, $u^{G(L)}_{il}$ and $u^{G(T)}_{il}$ denote the longitudinal wave part and the transverse wave part of the displacement Green's function. Considering the following elastodynamic radiation conditions (Eringen and Suhubi, 1975)

$$\lim_{R \to \infty} R(\sigma^{(L)}_{ij} n_j + \rho c_L \dot{u}^{(L)}_i) = 0, \tag{34}$$

$$\lim_{R \to \infty} R(\sigma^{(T)}_{ij} n_j + \rho c_T \dot{u}^{(T)}_i) = 0, \tag{35}$$

it is concluded that

$$\int_{S_R} I_{lk}\, dS = 0, \quad \text{as } R \to \infty. \tag{36}$$

In eqns (34) and (35), $u^{(L)}_i$ and $\sigma^{(L)}_{ij}$ represent the longitudinal wave part, while $u^{(T)}_i$ and $\sigma^{(T)}_{ij}$ represent the transverse wave part, of the scattered field. Equation (36) implies that the integral over S_R contributes nothing to the representation integral of the displacement gradients of the scattered field.

By substituting eqn (26) into Hooke's law the following representation formula for the traction components is obtained

$$t^{sc}_p(\mathbf{x}, t) = -C_{pqlk} n_q(\mathbf{x}) \int_S [(\sigma^{G}_{mnl} * u^{sc}_{m,n} + \rho u^{G}_{il} * \ddot{u}^{sc}_i)\delta_{jk} - \sigma^{G}_{ijl} * u^{sc}_{i,k} - u^{G}_{il,k} * \sigma^{sc}_{ij}] n_j\, dS, \mathbf{x} \notin S. \tag{37}$$

Application of eqn (37) to a 3D crack yields

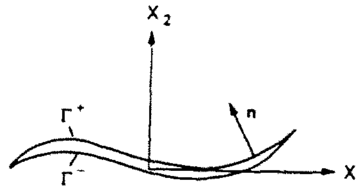

Fig. 3. A curved crack in a 2D geometry.

$$t_p^{sc}(\mathbf{x}, t) = -C_{pqlk} n_q(\mathbf{x}) \int_{A^+} [(\sigma^G_{mnl} * \Delta u_{m,n} + \rho u^G_{il} * \Delta \ddot{u}_i)\delta_{jk} - \sigma^G_{ijl} * \Delta u_{i,k}] n_j \, \mathrm{d}S,$$

$$\mathbf{x} \notin A^+, \tag{38}$$

where Δu_i are the crack-opening displacements and $\Delta u_{i,k}$ are their derivatives with respect to y_k. The last term of eqn (37) disappears because of the continuity of $\sigma^{sc}_{ij} n_j$ across the crack faces. By letting $\mathbf{x} \to A^+$ and by considering the boundary conditions on the faces of the crack, BIEs are obtained as

$$t_p^{in}(\mathbf{x}, t) = C_{pqlk} n_q(\mathbf{x}) \int_{A^+} [(\sigma^G_{mnl} * \Delta u_{m,n} + \rho u^G_{il} * \Delta \ddot{u}_i)\delta_{jk} - \sigma^G_{ijl} * \Delta u_{i,k}] n_j \, \mathrm{d}S,$$

$$\mathbf{x} \in A^+, \tag{39}$$

where t_p^{in} denotes the traction components on the faces of the crack induced by the incident wave, and the integral is understood in the sense of the Cauchy principal values. The system of BIEs (39) is valid for 3D cracks of arbitrary shapes. BIEs for 2D cracks in plane strain and anti-plane strain can be derived directly from eqn (39) by setting $\partial/\partial x_3 = 0$. This results in, for plane strain

$$t_\alpha^{in}(\mathbf{x}, t) = C_{\alpha\beta\gamma\delta} n_\beta(\mathbf{x}) \int_{\Gamma^+} [(\sigma^g_{\xi\eta\gamma} * \Delta u_{\xi,\eta} + \rho u^g_{\delta\gamma} * \Delta \ddot{u}_\delta)\delta_{\lambda\kappa} - \sigma^g_{\mu\lambda\gamma} * \Delta u_{\mu,\kappa}] n_\lambda \, \mathrm{d}s,$$

$$\mathbf{x} \in \Gamma^+, \tag{40}$$

while for anti-plane strain

$$t_3^{in}(\mathbf{x}, t) = \mu n_\beta(\mathbf{x}) \int_{\Gamma^+} [(\sigma^g_{3\alpha 3} * \Delta u_{3,\alpha} + \rho u^g_{33} * \Delta \ddot{u}_3)\delta_{\beta\gamma} - \sigma^g_{3\gamma 3} * \Delta u_{3,\beta}] n_\gamma \, \mathrm{d}s,$$

$$\mathbf{x} \in \Gamma^+, \tag{41}$$

where Γ^+ denotes the insonified side of the 2D crack (Fig. 3), and the superscript "g" indicates the 2D Green's functions (see Appendix A). Also here, the integrals are understood as the Cauchy principal values.

The unknown boundary quantities in the BIEs (39)–(41) are the crack-opening displacements and their derivatives, where the latter have the physical meaning of dislocation densities. An essential advantage of the new derivation presented here is that it immediately leads to non-hypersingular time-domain BIEs for crack analysis, and no elaborate manipulations such as integration by parts have been used. From this point of view, the new derivation allows an immediate and reliable numerical implementation. When the unknown quantities Δu_i and $\Delta u_{i,k}$ have been computed, stresses and strains at an arbitrary internal point can be calculated from eqn (26) and eqn (38). At internal points near the crack faces, i.e. for $\mathbf{x}$ close to A, accurate results for the stresses and strains can be obtained by using eqns (26) and (38). In the conventional formulation without regularization, considerable numerical errors may occur at such points because of the presence of hypersingularities. This phenomenon is often refered to as "boundary-layer effects". To calculate the displacement field at internal points the representation formula for the displacement, eqn (8), can be employed.

Clearly, BIEs can also be derived from other conservation integrals by following essentially the same procedure used here. Consider now for instance the complementary conservation integral to I_k which can be stated as

$$I_k^c = \int_S [\tfrac{1}{2}(\sigma_{mn} * u_{m,n} + \rho \ddot{u}_i * u_i)\delta_{jk} - \sigma_{ij,k} * u_i] n_j \, \mathrm{d}S - \int_V \rho f_{i,k} * u_i \, \mathrm{d}V = 0. \tag{42}$$

The assumptions for (42) to be valid are the same as for I_k. The proof of eqn (42) can be performed directly by using the divergence theorem, by considering eqns (1)–(4) and (6), and by employing the properties of Riemann convolutions (16a–d).

Following the procedure applied for deriving eqn (37), a novel representation integral for the scattered traction components is obtained from eqn (42). The result is

$$t_p^{sc}(\mathbf{x}, t) = -C_{pqlk} n_q(\mathbf{x}) \int_S [(\sigma_{mnl}^{G} * u_{m,n}^{sc} * + \rho u_{il}^{G} * \ddot{u}_i^{sc})\delta_{jk} - \sigma_{ijl,k}^{G} * u_i^{sc} - u_{il}^{G} * \sigma_{ij,k}^{sc}] n_j \, \mathrm{d}S,$$

$$\mathbf{x} \notin S. \tag{43}$$

By applying eqn (43) to the crack faces, by letting $\mathbf{x} \to A^+$ and by considering the boundary conditions (5), a new set of BIEs can be derived. However, these BIEs are again hypersingular and they have no advantages over those obtained by the conventional derivation. This implies that not every conservation integral will lead to a convenient BIE formulation for elastodynamic crack analysis.

4. EXAMPLES

The non-hypersingular time-domain BIEs (39) are valid for cracks of arbitrary shapes. The Laplace transform of (39) is identical to the BIEs obtained by Sládek and Sládek (1984), who used the conventional derivation in conjunction with a regularization technique. For a planar 3D crack located in the plane $x_3 = 0^{\pm}$ ($n_1 = n_2 = 0$, $n_3 = 1$) of an unbounded body, the BIEs (39) split into two uncoupled equations

$$\sigma_{33}^{in}(\mathbf{x}, t) = \int_{A^+} \{[(\lambda + 2\mu)\sigma_{\alpha 33}^{G} - \lambda \sigma_{33\alpha}^{G}] * \Delta u_{3,\alpha} + \rho(\lambda + 2\mu) u_{33}^{G} * \Delta \ddot{u}_3\} \, \mathrm{d}S,$$

$$\mathbf{x} \in A^+, \tag{44}$$

$$\sigma_{\beta 3}^{in}(\mathbf{x}, t) = \mu \int_{A^+} \{[\sigma_{\alpha\gamma\beta}^{G} - \sigma_{\alpha 33}^{G}\delta_{\beta\gamma}] * \Delta u_{\alpha,\gamma} + \rho u_{\alpha\beta}^{G} * \Delta \ddot{u}_\alpha\} \, \mathrm{d}S,$$

$$\mathbf{x} \in A^+, \quad \alpha = \beta = 1, 2, \tag{45}$$

in which σ_{33}^{in} and $\sigma_{\beta 3}^{in}$ are the stress components induced by incident waves. It should be noted here that eqn (44) is for the normal crack opening displacement Δu_3, while eqn (45) is for the transverse crack opening displacements Δu_α. The Fourier transform of eqns (44) and (45) yields the equations obtained by Budiansky and Rice (1979), via the conventional formulation in conjunction with regularization techniques.

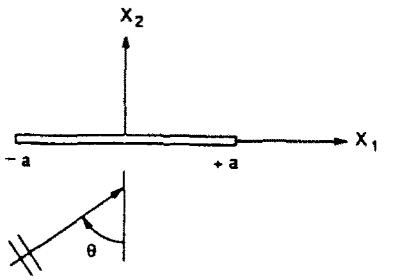

Fig. 4. A 2D straight crack.

For a 2D straight crack defined by $x_2 = 0^{\pm}$ and $|x_1| \leqslant a$ (Fig. 4), and for plane strain, the BIEs (40) take the following form

$$\sigma^{in}_{12}(\mathbf{x}, t) = \mu \int_{-a}^{a} \{[\sigma^{\xi}_{111} - \sigma^{\xi}_{122}] * \Delta u_{1,1} + \rho u^{\xi}_{11} * \Delta \ddot{u}_1\} \, dy_1, \quad |x_1| \leqslant a, \tag{46}$$

$$\sigma^{in}_{22}(\mathbf{x}, t) = \int_{-a}^{a} \{[(\lambda + 2\mu)\sigma^{\xi}_{112} - \lambda \sigma^{\xi}_{221}] * \Delta u_{2,1} + \rho(\lambda + 2\mu) u^{\xi}_{22} * \Delta \ddot{u}_2\} \, dy_1,$$

$$|x_1| \leqslant a, \tag{47}$$

while for anti-plane strain eqn (41) has the form

$$\sigma^{in}_{13}(\mathbf{x}, t) = \mu \int_{-a}^{a} (\sigma^{\xi}_{113} * \Delta u_{3,1} + \rho u^{\xi}_{33} * \Delta \ddot{u}_3) \, dy_1, \quad |x_1| \leqslant a. \tag{48}$$

In the case of plane strain, the BIE for the normal crack opening displacement Δu_2 decouples from the one for the transverse crack opening displacement Δu_1, as can be seen from eqns (46) and (47). In the frequency domain, the system of BIEs (46) and (47) has the same form as those derived by Tan (1975) who used the conventional formulation in conjunction with regularization.

In general, the BIEs presented here must be solved numerically. Higher-order shape functions for the unknown crack opening displacements are desirable since the BIEs in the new derivation contain not only the functions Δu_i, but also their derivatives with respect to space and time variables. Special care must be taken in the numerical implementation to account for the local behavior of Δu_i and $\Delta u_{i,j}$ near crack edges, and for the singularities of the Green's functions at $\mathbf{x} = \mathbf{y}$. Because the BIEs presented here are time dependent, the discretization of t is necessary. For the 3D case, Riemann convolutions of the BIEs can be evaluated analytically by using the sifting property of the delta function. In the case of plane strain and anti-plane strain, time integrations must be generally carried out numerically, but it is also possible to perform time integrations analytically with linear interpolation functions in t (Guo *et al.*, 1988; Hirose and Achenbach, 1988, 1989; Nishimura *et al.*, 1987a,b, 1988; Zhang and Achenbach, 1988a, 1989a; Zhang, 1990a). Spatial integrations can be performed numerically for regular elements ($\mathbf{x} \neq \mathbf{y}$) by using suitable quadrature formula, while careful analytical treatments for singular elements ($\mathbf{x} = \mathbf{y}$) are recommended.

As a test example, a straight anti-plane crack has been chosen because of its simplicity. In this case, the BIE is given by eqn (48). Here, σ^{in}_{32} is selected as a plane impulse of the form

$$\sigma_{32}^{in} = \tau_0 \cos\theta\, H[c_T t - \sin\theta(x_1 + a) - \cos\theta\, x_2], \tag{49}$$

where τ_0 is the amplitude, θ is the angle of incidence of the incident wave, and $H[*]$ is the Heaviside function. The unknown function $\Delta u_3(y_1, \tau)$ is approximated by

$$\Delta u_3(y_1, \tau) = \sum_{j=1}^{J} \sum_{n=1}^{N} \mu_j(y_1)\eta^n(\tau)(\Delta u_3)_j^n, \tag{50}$$

in which $\mu_j(y_1)$ is taken to be unity over each element except for elements near crack tips. For these elements a special function

$$\mu_j(y_1) = (a \pm y_1)^{1/2} \tag{51}$$

is applied to describe the proper behavior of Δu_3 at crack tips with $y_1 = \pm a$. A piecewise linear shape function is used for $\eta^n(\tau)$

$$\eta^n(\tau) = \begin{cases} 1 - |\tau - n\Delta t|/\Delta t, & |\tau - n\Delta t| \leqslant \Delta t, \\ 0, & \text{otherwise.} \end{cases} \tag{52}$$

With eqn (52) the convolution integrals of (48) have been carried out analytically for each time interval $[t_{n-1}, t_n]$. Spatial integrations of eqn (48) have been performed analytically for a constant shape function and numerically for the "crack tip shape function", eqn (51). The Mode-III stress intensity factor can be calculated by using the relation

$$K_{III}^{\pm}(t) = \frac{\mu(2\pi)^{1/2}}{4} \lim_{x_1 \to \pm a} \frac{1}{(a \mp x_1)^{1/2}} \Delta u_3(x_1, t), \tag{53}$$

where "$\pm$" indicates the crack tips at $x_1 = \pm a$.

A total number of 50 elements of equal size, and 200 time steps have been used in the numerical calculations. The time increment is chosen as $c_T \Delta t = 0.4a$. No instability sign is noted in the computational procedure. The calculated dynamic stress intensity factors, which are normalized by their corresponding static values, are shown in Figs 5–6 for comparison with the exact results of Thau and Lu (1970). A very good agreement between both results is obtained.

5. CONCLUSIONS

A novel application of conservation integrals is presented, to derive non-hypersingular time-domain BIEs for transient elastodynamic crack analysis. The new derivation is based on an elastodynamic conservation integral, the I_k integral. Boundary integral equations follow from I_k in a direct and natural way, and they are immediately non-hypersingular. This is an important advantage for the development of a numerical procedure for solving these BIEs and for an accurate calculation of the stresses and strains at internal points close to the crack faces. The BIEs presented here are valid for 3D or 2D cracks of arbitrary shapes, and their Laplace and Fourier transforms agree with the known results obtained by other authors using the conventional derivation in conjunction with regularization techniques. The unknown quantities of the BIEs in the new derivation are the crack-opening displacements and their derivatives with respect to space and time variables. Numerical examples have shown that highly accurate results for elastodynamic stress intensity factors can be obtained. BIEs for general initial boundary value problems (not necessary crack problems) can be derived from the representation integral for the traction components, eqn (37). For the static case, the advantages and drawbacks of this kind of formulation compared to the conventional formulation have been discussed by Hu (1987) by using a similar procedure to that in this paper, and by Okada *et al.* (1988) who presented a

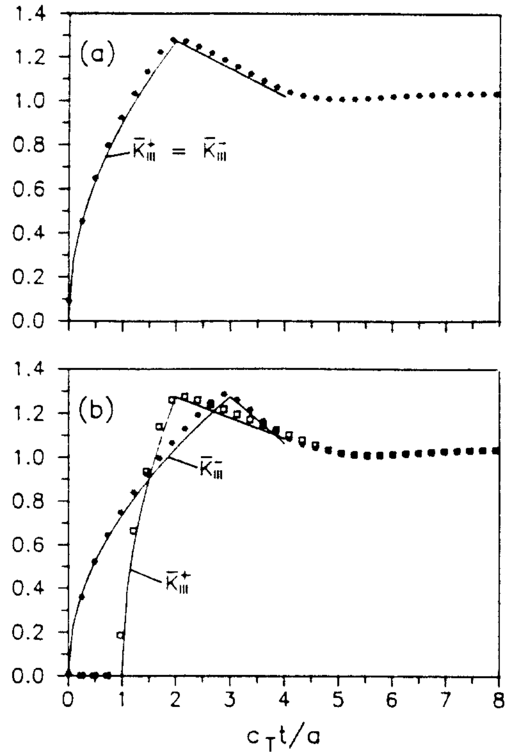

Fig. 5. Normalized dynamic stress intensity factors, ——: exact results, $\bullet\ \bullet\ \bullet$, □ □ □: results of this paper; (a) $\theta = 0°$, (b) $\theta = 30°$.

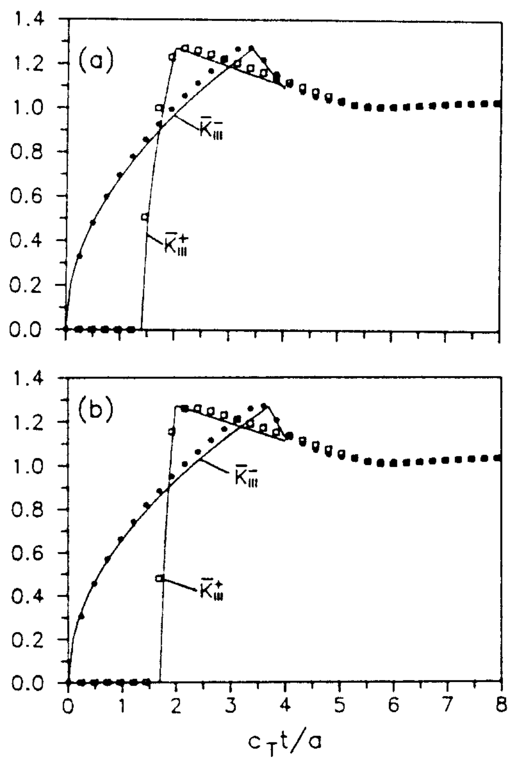

Fig. 6. Normalized dynamic stress intensity factors, ——: exact results, $\bullet\ \bullet\ \bullet$, □ □ □: results of this paper; (a) $\theta = 45°$, (b) $\theta = 60°$.

displacement gradient BIE based on a weak form of the linear momentum balance law of elastostatics.

Finally, it should be noted that BIEs can also be derived from other conservation integrals by following essentially the same procedure as presented in this paper. This does not, however, imply that every conservation integral will lead to a convenient BIE formulation for elastodynamic crack analysis. The complementary conservation integral to I_k gives rise, for example, to hypersingular BIEs which offer no advantages over the conventional derivation.

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APPENDIX A: GREEN'S FUNCTION

The Green's function for 3D elastodynamic states is given by (Eringen and Suhubi, 1975)

$$u_{ij}^{G(L)}(\mathbf{x}, t; \mathbf{y}, 0) = \frac{1}{4\pi\rho}\left[-\left(\frac{3r_ir_j}{r^3} - \frac{\delta_{ij}}{r}\right)\int_0^{c_L^{-1}} \lambda\delta(t-\lambda r)\,d\lambda + \frac{r_ir_j}{c_L^2 r^3}\delta(t-r/c_L)\right], \tag{A1}$$

$$u_{ij}^{G(T)}(\mathbf{x}, t; \mathbf{y}, 0) = \frac{1}{4\pi\rho}\left[\left(\frac{3r_ir_j}{r^3} - \frac{\delta_{ij}}{r}\right)\int_0^{c_T^{-1}} \lambda\delta(t-\lambda r)\,d\lambda - \frac{r_ir_j}{c_T^2 r^3}\delta(t-r/c_T) + \frac{\delta_{ij}}{c_T^2 r}\delta(t-r/c_T)\right], \tag{A2}$$

$$u_{ij}^{G} = u_{ij}^{G(L)} + u_{ij}^{G(T)}, \tag{A3}$$

where $r = |\mathbf{x}-\mathbf{y}|$, $r_i = x_i - y_i$, and $u_{ij}^{G(L)}$ and $u_{ij}^{G(T)}$ represent the longitudinal and the transverse wave part of the displacement field u_{ij}^{G}, respectively. Here, the index "i" indicates the direction of the displacements at an observation point $\mathbf{x}$, while "j" defines the direction of the applied impulsive unit point force at a point $\mathbf{y}$. Similarly, the stress Green's function can be partitioned as

$$\sigma_{ijk}^{G} = \sigma_{ijk}^{G(L)} + \sigma_{ijk}^{G(T)}, \tag{A4}$$

in which $\sigma_{ijk}^{G(L)}$ and $\sigma_{ijk}^{G(T)}$ are the stress components corresponding to the longitudinal and the transverse wave parts of the displacement field. Expressions for $\sigma_{ijk}^{G(L)}$ and $\sigma_{ijk}^{G(T)}$ can be obtained by substituting (A1) and (A2) into Hooke's law, but they will not be given here for the sake of brevity.

The Green's functions for 2D plane strain and anti-plane strain elastodynamic states are

$$u_{\alpha\beta}^{g}(\mathbf{x}, t; \mathbf{y}, 0) = \frac{1}{2\pi\rho}\left\{\frac{1}{c_L r^2} H(c_L t - r)\left[\frac{2c_L^2 t^2 - r^2}{R_L} r_{,\alpha} r_{,\beta} - R_L \delta_{\alpha\beta}\right] - \frac{1}{c_T r^2} H(c_T t - r)\left[\frac{2c_T^2 t^2 - r^2}{R_T} r_{,\alpha} r_{,\beta} - \frac{c_T^2 t^2 - r^2}{R_T}\delta_{\alpha\beta}\right]\right\}, \tag{A5}$$

$$u_{33}^{g}(\mathbf{x}, t; \mathbf{y}, 0) = \frac{1}{2\pi\mu}\frac{H(t - r/c_T)}{(t^2 - r^2/c_T^2)^{1/2}}, \tag{A6}$$

where

$$R_\xi = (c_\xi^2 t^2 - r^2)^{1/2}, \quad \xi = L, T, \tag{A7}$$

and $H[\bullet]$ denotes the Heaviside step function. The corresponding stress components can be obtained from (A5), (A6) and Hooke's law.

APPENDIX B: ASYMPTOTICS OF THE 3-D GREEN'S FUNCTION

In deriving eqn (36), asymptotic expressions of the Green's functions have been used. By using the relations

$$r_{,i} = n_i, \tag{B1}$$

$$t_{ik}^{G} = \sigma_{ijk}^{G} n_j, \tag{B2}$$

one obtains

$$t_{ik}^{G(L)} = -\rho c_L \dot{u}_{ik}^{G(L)} - 6\rho c_L^2 u_{ik}^{G(L)}/r + \frac{1}{4\pi r^2}\left\{[2n_in_k - (1-2c_T^2/c_L^2)\delta_{ik}]\cdot\delta(t-r/c_L) + c_L(3n_in_k - \delta_{ik})\int_0^{c_L^{-1}} \delta(t-\lambda r)\,d\lambda\right\}, \tag{B3}$$

$$t_{ik}^{G(T)} = -\rho c_T \dot{u}_{ik}^{G(T)} - 6\rho c_T^2 u_{ik}^{G(T)}/r - \frac{1}{4\pi r^2}\left\{2(n_i n_k - 2\delta_{ik})\delta(t - r/c_T) - c_T(3n_i n_k - \delta_{ik})\int_0^{c_T^{-1}} \delta(t-\lambda r)\,\mathrm{d}\lambda\right\}, \tag{B4}$$

where n_i is the unit outward normal vector to S_R, and $t_{ik}^{G(L)}$ and $t_{ik}^{G(T)}$ are the traction components on S_R corresponding to the longitudinal and the transverse wave part of the displacement Green's function. For $r \to \infty$, the following asymptotics are obtained from (B3) and (B4) as

$$t_{ik}^{G(L)} = -\rho c_L \dot{u}_{ik}^{G(L)} + O(1/r^2), \tag{B5}$$

$$t_{ik}^{G(T)} = -\rho c_T \dot{u}_{ik}^{G(T)} + O(1/r^2), \tag{B6}$$

or in equivalent forms as

$$t_{ik}^{G} = t_{ik}^{G(L)} + t_{ik}^{G(T)} = -\rho \sum_{\zeta = L,T} c_\zeta \dot{u}_{ik}^{G(\zeta)} + O(1/r^2). \tag{B7}$$

Moreover, it can be easily shown that for $r \to \infty$

$$u_{il,k}^{G(L)} = \frac{-1}{c_L}\dot{u}_{il}^{G(L)} n_k + O(1/r^2), \tag{B8}$$

$$u_{il,k}^{G(T)} = \frac{-1}{c_T}\dot{u}_{il}^{G(T)} n_k + O(1/r^2), \tag{B9}$$

or

$$u_{il,k}^{G} = \sum_{\zeta = L,T} \frac{-1}{c_\zeta}\dot{u}_{il}^{G(\zeta)} n_k + O(1/r^2). \tag{B10}$$